Singular Solutions for the Conformal Dirac-Einstein Problem on the Sphere

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Abstract In this paper we construct singular solutions to the conformal Dirac-Einstein system on the 3-sphere; in particular, we mainly focus on solutions admitting singularities at two points.

Keywords: Dirac-Einstein problem; Singular solutions.

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1 Introduction and main results

Let $(\mathbb{S}^3, g_s, \Sigma\mathbb{S}^3)$ be the unit sphere of dimension three equipped with its standard metric g_s and its canonical spin bundle $\Sigma\mathbb{S}^3$. For $(u,\psi) \in C^\infty(\mathbb{S}^3) \times C^\infty(\Sigma\mathbb{S}^3)$, let us consider the following coupled system:

$$\begin{cases} L_{g_s} u = |\psi|^2 u \\ D_{g_s}\psi = |u|^2\psi \end{cases} \text{on } \mathbb{S}^3, \tag{1}$$

where $L_{g_s} = -\Delta_{g_s} + \frac{R_{g_s}}{8} = -\Delta_{g_s} + \frac{3}{4}$ and D_{g_s} are the conformal Laplacian on the sphere and the standard Dirac operator, respectively; $-\Delta_{g_s}$ is the Laplace-Beltrami operator and $R_{g_s} = 6$ is the scalar curvature of the unit sphere. This is the critical points equation of the Dirac-Einstein functional

$$E(u,\psi) = \frac{1}{2}\int_{\mathbb{S}^3} |\nabla u|^2 + \frac{R_{g_s}}{8}u^2 + \langle D_{g_s}\psi, \psi\rangle - |\psi|^2|u|^2 dv_{g_s},$$

here $\langle\cdot,\cdot\rangle$ denotes the compatible Hermitian metric on $\Sigma\mathbb{S}^3$ (we will give the precise definitions in the next section). Now, by using the stereographic projection, which is a

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conformal map from $\mathbb{S}^3$ to $\mathbb{R}^3$ with its standard Euclidean metric, the system (1) turns into

$$\begin{cases} -\Delta u = |\psi|^2 u \\ \\ D\psi = |u|^2\psi \end{cases} \text{on } \mathbb{R}^3, \tag{2}$$

We are interested in finding singular solutions of (1) on $\mathbb{S}^3 \setminus \Lambda$, where Λ is a pair of antipodal points on $\mathbb{S}^3$, that is we are looking for solutions of

$$\begin{cases} L_{g_s} u = |\psi|^2 u \\ \\ D_{g_s}\psi = |u|^2\psi \end{cases} \text{on } \mathbb{S}^3 \setminus \Lambda, \tag{3}$$

in the distributional sense, with u singular on Λ. This can be seen as a coupled singular Yamabe and spinorial Yamabe problem (see for instance [17, 18, 15, 12, 13, 14] and the references therein for a detailed description of the singular Yamabe type problems). Again, by means of the stereographic projection, the system (3) turns into

$$\begin{cases} -\Delta u = |\psi|^2 u \\ \\ D\psi = |u|^2\psi \end{cases} \text{on } \mathbb{R}^3 \setminus \{0\}, \tag{4}$$

One can approach the previous equations in two different ways, which however lead to the same system. The first one is more geometric and it starts by noticing that $\mathbb{R}^3 \setminus \{0\}$ is conformal to $\mathbb{R} \times \mathbb{S}^2$ via the conformal map $(r, \theta) \to (-\ln(r), \theta) = (t, \theta)$. Now, using the conformal invariance of the Laplacian and the Dirac operator, the problem (4) becomes:

$$\begin{cases} L_g w = |\phi|^2 w \\ \\ D_g\phi = w^2\phi \end{cases} \text{on } \mathbb{R} \times \mathbb{S}^2, \tag{5}$$

where $L_g = -\Delta_{t,\theta} + \frac{1}{4}$ and D_{g_s} are the conformal Laplacian and the Dirac operator on $\mathbb{R} \times \mathbb{S}^2$ equipped with the canonical product metric $g = dt^2 + d\theta^2$. Inspired by [2, 19] and the spin representation on a product manifold, we let $\phi = \psi \otimes \varphi$ where φ is a Killing spinor on $\mathbb{S}^2$. Then we have that φ is an eigenspinor for the Dirac operator on $\mathbb{S}^2$ with eigenvalue 1 and since $|\varphi|$ is constant, we assume it to be 1; also, we will look for solution of the form $w(t, \theta) = u(t)$. Then, by taking into account the natural splitting of the spin bundle, one gets the following system for $\psi = \psi^+ \oplus \psi^-$:

$$\begin{cases} -u'' + \frac{1}{4}u = (|\psi^+|^2 + |\psi^-|^2)u \\ \\ i\dfrac{d\psi^+}{dt} + i\psi^- = u^2\psi^+ \\ \\ i\dfrac{d\psi^-}{dt} + i\psi^+ = u^2\psi^- \end{cases} \tag{6}$$

Since ψ is a complex valued function, if we put $\psi^+ = a + ib$ and $\psi^- = a - ib$ in the previous system, we get the following one:

$$\begin{cases} u'' = -(a^2+b^2)u + \frac{1}{4}u \\ a' = -a + u^2 b \\ b' = b - u^2 a \end{cases} \tag{7}$$

The second method is more analytic. It was used in several works as an ansatz to find particular solutions (we refer the reader to [6, 16, 20, 12] and the references therein). We start by defining the space of "radial" spinors $E(\mathbb{R}^3)$ as follows:

$$E(\mathbb{R}^3) = \left\{ \psi(x) = f_1(|x|)\gamma_0 + \frac{f_2(|x|)}{|x|} x \cdot \gamma_0 \,; x \in \mathbb{R}^3, f_1, f_2 \in C^\infty(0,\infty), \gamma_0 \in \mathbb{S}^2_{\mathbb{C}} \right\},$$

where $\cdot$ stands for the Clifford multiplication and $\mathbb{S}^2_{\mathbb{C}}$ denotes the complex unit sphere in $\mathbb{C}^2$; we notice that this space is stable under the action of the Dirac operator. This second approach relies on the ansatz that $u(x) = u(|x|)$ and $\psi \in E(\mathbb{R}^3)$: so, if in (4) we apply the Emden-Fowler change of variable $r = e^{-t}$ and write $f_1(r) = -a(t)e^t$, $f_2(r) = b(t)e^t$, we obtain again the system (7).
Now, we see that the system (7) can be viewed as a Hamiltonian system

$$\begin{cases} a' = -a + u^2 b \\ b' = b - u^2 a \\ u' = v \\ v' = -\left(a^2 + b^2 - \frac{1}{4}\right) u \end{cases} \tag{8}$$

where the Hamiltonian function H is given by:

$$\begin{aligned} H(a,b,u,v) &= \frac{v^2}{2} + \frac{u^2}{2}\left(a^2+b^2-\frac{1}{4}\right) - ab \\ &= \frac{v^2}{2} + \frac{(u^2-1)}{2}\left(a^2+b^2-\frac{1}{4}\right) + \frac{1}{2}\left((a-b)^2 - \frac{1}{4}\right), \end{aligned}$$

in fact,

$$\begin{cases} \dot{a} = \dfrac{\partial H}{\partial b}(a,b,u,v) \\[2ex] \dot{b} = -\dfrac{\partial H}{\partial a}(a,b,u,v) \\[2ex] \dot{u} = \dfrac{\partial H}{\partial v}(a,b,u,v) \\[2ex] \dot{v} = -\dfrac{\partial H}{\partial u}(a,b,u,v). \end{cases}$$

The equilibrium points, with $u \geq 0$, of the system are

$$P_0 = (0,0,0,0), \qquad P^\pm = \left(\pm\sqrt{\frac{1}{8}}, \pm\sqrt{\frac{1}{8}}, 1, 0\right)$$

and they correspond to the energy levels $H = 0$ and $H = -\frac{1}{8}$; in particular P_0 is a saddle point and $P^{\pm}$ are center points. From the analysis of this Hamiltonian system, we have the following:

Theorem 1.1. *Let $T_0 = 2^{\frac{3}{4}}\pi$. Then, for $T \in (T_0, \infty)$, there exists a sequence of $2T$-periodic solutions (u_T, ψ_T) to (6). Moreover,*

(i) there exist $T_1 < T_2$ such that for $T \in (T_0, T_1) \cup (T_2, +\infty)$, the family is different from the constant equilibrium solutions;

(ii) when $T \to T_0$,

$$\left(u_T, |\psi_T|^2\right) \to \left(1, \frac{1}{4}\right), \quad \text{in } C^{2,\alpha}_{loc}(\mathbb{R}) \times C^{1,\beta}_{loc}(\mathbb{R}),\ 0 < \alpha, \beta < 1;$$

(iii) when $T \to \infty$, there exists t_0 such that

$$(u_T, \psi_T) \to \left(u_0(\cdot - t_0), \psi_0(\cdot - t_0)\right), \quad \text{in } C^{2,\alpha}_{loc}(\mathbb{R}) \times C^{1,\beta}_{loc}(\mathbb{R}),\ 0 < \alpha, \beta < 1.$$

where (u_0, ψ_0) is the solution of (6) given by the nontrivial homoclinic of (8).

In terms of singular solutions of the conformal Dirac-Einstein equation (2) on $\mathbb{R}^3 \setminus \{0\}$, which is equivalent to the problem (3) on $\mathbb{S}^3 \setminus \Lambda$, we obtain

Corollary 1.1. *For $T > T_2 > 0$, there exists $\lambda > 0$ and a one parameter family (u_T, ψ_T) of singular solutions of problem (4) such that, when $T \to \infty$*

$$(u_T, \psi_T) \to (U_\lambda, \Psi_\lambda), \quad \text{in } C^{2,\alpha}_{loc}(\mathbb{R}^3 \setminus \{0\}) \times C^{1,\beta}_{loc}(\Sigma\mathbb{R}^3 \setminus \{0\}),\ 0 < \alpha, \beta < 1,$$

where $(U_\lambda, \Psi_\lambda)$ is a ground state solution of (2).

2 Geometric and analytic settings

Here we briefly recall some notations and properties of the relevant operators that we are going to use. On a general compact, without boundary, three dimensional Riemannian manifold (M, g), we consider the conformal Laplacian acting on functions by

$$L_g u := -\Delta_g u + \frac{1}{8} R_g u,$$

where Δ_g is the standard Laplace-Beltrami operator and R_g is the scalar curvature. Then the conformal invariance of L_g means that if $\tilde{g} = g_u = u^4 g$ is a metric in the conformal class of g, then we have $L_{\tilde{g}} f = u^{-5} L_g(uf)$. We will denote by $H^1(M)$ the usual Sobolev space on M, and we recall that by the Sobolev embedding theorems there is a continuous embedding

$$H^1(M) \hookrightarrow L^p(M), \quad 1 \leq p \leq 6,$$

which is compact if $1 \leq p < 6$.
Regarding the spinorial part, we denote by ΣM the canonical spinor bundle associated to M (see for instance [7]), whose sections are called spinors. On this bundle one defines a natural Clifford multiplication

$$\text{Cliff} : C^\infty(TM \otimes \Sigma M) \longrightarrow C^\infty(\Sigma M),$$

a hermitian metric $\langle \cdot, \cdot \rangle$, and a natural metric connection

$$\nabla^\Sigma : C^\infty(\Sigma M) \longrightarrow C^\infty(T^*M \otimes \Sigma M).$$

Therefore the Dirac operator D_g acting on spinors is given by the composition

$$D_g : C^\infty(\Sigma M) \longrightarrow C^\infty(\Sigma M), \quad D_g = \text{Cliff} \circ \nabla^\Sigma,$$

where $T^*M \simeq TM$ are identified by means of the metric; the conformal invariance for the Dirac operator reads as follows: if $\tilde{g} = u^4 g$, then $D_{\tilde{g}}\psi = u^{-4} D_g(u^2\psi)$.
Next we recall that the Dirac operator D_g on a compact manifold is self-adjoint in $L^2(\Sigma M)$, has compact resolvent and there exists a complete L^2-orthonormal basis of eigenspinors $\{\psi_i\}_{i\in\mathbb{Z}}$ satisfying

$$D_g \psi_i = \lambda_i \psi_i,$$

where the eigenvalues $\{\lambda_i\}_{i\in\mathbb{Z}}$ are unbounded, that is $|\lambda_i| \to \infty$, as $|i| \to \infty$. For a given $\psi \in L^2(\Sigma M)$, written as $\psi = \sum_{i\in\mathbb{Z}} a_i\psi_i$, we define the operator

$$|D_g|^s : L^2(\Sigma M) \to L^2(\Sigma M), \quad |D_g|^s(\psi) = \sum_{i\in\mathbb{Z}} a_i |\lambda_i|^s \psi_i.$$

Hence we define the space $H^s(\Sigma M)$ as the natural domain of $|D_g|^s$, that is

$$\psi \in H^s(\Sigma M) \iff \sum_{i\in\mathbb{Z}} a_i^2 |\lambda_i|^{2s} < +\infty.$$

It holds that $H^s(\Sigma M)$ coincides with the Sobolev space $W^{s,2}(\Sigma M)$, and moreover one can define the inner product

$$\langle u, v\rangle_s = \langle |D_g|^s u, |D_g|^s v\rangle_{L^2},$$

which induces an equivalent norm in $H^s(\Sigma M)$; in particular, for $s = \frac{1}{2}$, we will consider

$$\langle u, u\rangle := \langle u, u\rangle_{\frac{1}{2}} = \|u\|^2$$

as norm for the space $H^{\frac{1}{2}}(\Sigma M)$. For this space, there exists the following continuous embedding

$$H^{\frac{1}{2}}(\Sigma M) \hookrightarrow L^p(\Sigma M), \quad 1 \leq p \leq 3,$$

which is compact if $1 \leq p < 3$. Now, given the L^2-orthonormal basis of eigenspinors $\{\psi_i\}_{i\in\mathbb{Z}}$, we will denote by ψ_i^- the eigenspinors with negative eigenvalue, ψ_i^+ the eigenspinors with positive eigenvalue and ψ_i^0 the eigenspinors with zero eigenvalue and we recall that the kernel of D_g is finite dimensional. Therefore we set

$$H^{\frac{1}{2},-} := \overline{\text{span}\{\psi_i^-\}_{i\in\mathbb{Z}}}, \quad H^{\frac{1}{2},0} := \text{span}\{\psi_i^0\}_{i\in\mathbb{Z}}, \quad H^{\frac{1}{2},+} := \overline{\text{span}\{\psi_i^+\}_{i\in\mathbb{Z}}},$$

where the closure is with respect to the topology induced by the previous norm, and we have the splitting:

$$H^{\frac{1}{2}}(\Sigma M) = H^{\frac{1}{2},-} \oplus H^{\frac{1}{2},0} \oplus H^{\frac{1}{2},+}, \tag{9}$$

and we will denote by P^+ and P^- be the projectors on $H^{\frac{1}{2},+}$ and $H^{\frac{1}{2},-}$ respectively. Finally we recall a regularity results for weak solutions, (see [11], Theorem 3.1): if $(u,\psi) \in H^1(M) \times H^{\frac{1}{2}}(\Sigma M)$ is a weak solution of the system of equations (1), on a closed three dimensional spin manifold $(M, g, \Sigma M)$, then $(u,\psi) \in C^{2,\alpha}(M) \times C^{1,\beta}(\Sigma M)$, for some $0 < \alpha, \beta < 1$.

3 Proof of the main theorem

Here we will prove the main result, Theorem 1.1. In particular we will start from the existence of periodic orbits of (8) near the equilibrium points.

3.1 Small oscillation and periodic orbits

First of all we transform the system (8) using the auxiliary variables $\bar{a} = \frac{a+b}{\sqrt{2}}$ and $\bar{b} = \frac{a-b}{\sqrt{2}}$. So the system becomes:

$$\begin{cases} \bar{a}' = -(1+u^2)\bar{b} \\ \bar{b}' = (u^2-1)\bar{a} \\ u' = v \\ v' = \left(\frac{1}{4} - (\bar{a}^2 + \bar{b}^2)\right) u \end{cases} \tag{10}$$

The new Hamiltonian is then

$$\bar{H}(\bar{a}, \bar{b}, u, v) = \frac{1}{2}v^2 + \frac{1}{2}(u^2-1)\left(\bar{a}^2 + \bar{b}^2 - \frac{1}{4}\right) + \frac{1}{2}\left(2\bar{b}^2 - \frac{1}{4}\right).$$

The equilibrium points are then $(0,0,0,0)$ and $(\pm\frac{1}{2}, 0, 1, 0)$. The linearization of the right hand side of (10) at $(\pm\frac{1}{2}, 0, 1, 0)$ leads to the following matrix

$$C = \begin{bmatrix} 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix}.$$

In particular, C has two real eigenvalues $\pm 2^{\frac{1}{4}}$ and two complex eigenvalues $\pm i 2^{\frac{1}{4}}$. Therefore, one can apply the Lyapunov's center theorem, in order to prove the existence of a positive δ and a family $(x_r)_{r\in(-\delta,\delta)} := (a_r, b_r, u_r, v_r)$ of periodic solutions with period T_r starting from the equilibrium point $(\frac{1}{2}, 0, 1, 0)$ where $T_0 = \frac{2\pi}{2^{\frac{1}{4}}}$.

3.2 Existence of large period solutions

In this section we will denote by $H^1_{per}(T) = H^1_{per}([-T,T];\mathbb{R})$, the Sobolev space of $2T$-periodic functions endowed with the equivalent norm

$$\|u\|^2 = \|u\|^2_{H^1} = \int_{-T}^{T} |u'|^2 + \frac{1}{4}u^2 dt.$$

Also, if $z(t) = a(t) + ib(t) \simeq \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}$, for some $2T$-periodic scalar functions a, b, we will denote with A the operator defined by $Az = -Jz' + JBz$, where

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

We notice that A is self-adjoint since $JB = -BJ$ and moreover $A^2 z = -z'' + z$. Therefore we consider the Sobolev space of $2T$-periodic (complex valued) functions $H^{\frac{1}{2}}_{per}(T) = H^{\frac{1}{2}}_{per}([-T,T];\mathbb{C})$ and we use the norm given by

$$\|z\|^2 = \|z\|^2_{H^{\frac{1}{2}}} = \int_{-T}^{T} \left(|A|^{\frac{1}{2}}|z|\right)^2 dt.$$

Now let us consider the functional

$$E : H^1_{per}(T) \times H^{\frac{1}{2}}_{per}(T),$$

$$E(u,z) = \frac{1}{2}\int_{-T}^{T} |u'|^2 + \frac{1}{4}u^2 + \langle Az, z\rangle dt - \frac{1}{2}\int_{-T}^{T} u^2|z|^2 dt.$$

In particular, a direct computation shows that critical points of E solve the system (7).

Lemma 3.1. *The functional E satisfies the Palais-Smale condition (PS).*

Proof. The idea is very similar to the proof of the PS condition for the Dirac-Einstein equation as in [11]. So we consider a (PS) sequence $(u_n, z_n) \in H^1_{per}(T) \times H^{\frac{1}{2}}_{per}(T)$ at the level $c \in \mathbb{R}$. Thus

$$\int_{-T}^{T} |u_n'|^2 + \frac{1}{4}u_n^2 \; dt + \langle Az_n, z_n\rangle dt - \int_{-T}^{T} u_n^2|z_n|^2 \; dt \to 2c \tag{11}$$

and

$$\begin{cases} -u_n'' + \frac{1}{4}u_n = u_n|z_n|^2 + o(1) \\ \\ Az_n = u_n^2 z_n + o(1) \end{cases} \tag{12}$$

Multiplying the first equation of (12) by u_n and the second equation of (12) by z_n and substituting it in (11) we have

$$\|u_n\|^2 = 2c + o(\|z_n\|)$$

and

$$\int_{[-T,T]} u_n^2 |z_n|^2 \ dt = 2c + o(\|u_n\| + \|z_n\|).$$

Moreover, we have that

$$\begin{aligned}
\|z_n^+\|^2 &= \int_{[-T,T]} u_n^2 \langle z_n, z_n^+ \rangle \ dt + o(\|z_n\|) \\
&\le \left(\int_{[-T,T]} u_n^2 |z_n|^2 dt \right)^{\frac{1}{2}} \left(\int_{[-T,T]} u_n^2 |z_n^+|^2 dt \right)^{\frac{1}{2}} + o(\|z_n\|) \qquad (13) \\
&\le C \left(2c + o(\|u_n\| + \|z_n\|)\right) \|z_n^+\| + o(\|z_n\|).
\end{aligned}$$

Similarly,

$$\|z_n^-\|^2 \le C \left(2c + o(\|u_n\| + \|z_n\|)\right) \|z_n^+\| + o(\|z_n\|).$$

Thus, $\|z_n\|$ and $\|u_n\|$ are bounded. So up to a subsequence, $u_n \to u_\infty$ weakly in $H^1_{per}(T)$ and strongly in $C^{0,\frac{1}{2}}$ and $z_n \to z_\infty$ weakly in $H^{\frac{1}{2}}_{per}(T)$ and strongly in L^p for all $1 \le p < \infty$. So to finish, we write

$$\|u_n\|^2 = \int_{[-T,T]} u_n^2 |z_n|^2 \ dt$$

But $u_n \to u_\infty$ in L^∞ and $z_n \to z_\infty$ in L^2, therefore u_n converges strongly to u_∞ in $H^1_{per}(T)$; a similar argument works for (z_n) and hence, this finish the proof of the (PS) condition for E. □

We have the following

Proposition 3.1. *Let us consider the natural splitting $H^{\frac{1}{2}}_{per}(T) = H^- \oplus H^+$ (as in 9). Then there exists a functional $g : H^1_{per}(T) \times H^+ \to H^-$ satisfying, for $v \in H^+$*

$$E(u, v + w) < E(u, v + g(u, v)), \ \textit{for all } w \in H^-, w \ne g(u, v). \qquad (14)$$

Proof. We first notice that

$$\begin{aligned}
E(u, v + w) &= \frac{1}{2}\Big(\|u\|^2 + \|v\|^2 - \|w\|^2 - \int_{-T}^{T} u^2 |v + w|^2 \ dt \Big) \\
&= \frac{1}{2} \left(\|u\|^2 + \|v\|^2 - \int_{-T}^{T} u^2 |v|^2 dt \right) + K(w),
\end{aligned}$$

where $K : H^- \to \mathbb{R}$ is defined by

$$K(w) = -\|w\|^2 - \int_{-T}^{T} u^2 |w|^2 \ dt - 2 \int_{-T}^{T} u^2 \langle v, w \rangle \ dt$$

is strictly concave and anti-coercive, so it has a unique maximizer $w_0 = g(u, v)$. This maximizer satisfies $Dw_0 = P^-(u^2(w_0 + v))$, with P^- the projector on H^-, therefore it satisfies property (14). □

Lemma 3.2. *Let $(u,v) \in H^1_{per}(T) \times H^+$ and g given by the previous Proposition. Let us define $F(u,v) = E(u, v + g(u,v))$. Then F has the mountain pass geometry, namely we have*

(i) $F(0) = 0$

(ii) There exists $r > 0$ such that if $\|u\|^2 + \|v\|^2 \le r$, then $F(u,v) \ge 0$; in particular if $\|u\|^2 + \|v\|^2 = r$ then $F(u,v) \ge \alpha = \alpha(r) > 0$.

(iii) If $\int_{-T}^{T} u^2 |v|^2 \, dt \neq 0$, there exist $t, s > 0$ such that $F(tu, sv) < 0$.

(iv) The functional F satisfy the PS condition.

Proof. Regarding (i), we notice that $g(0,0) = 0$, hence $F(0) = 0$. Next, we notice that

$$
\begin{aligned}
F(u,v) &\ge \frac{1}{2}\left(\|u\|^2 + \|v\|^2 - \int_{-T}^{T} u^2 |v|^2 \, dt\right) \\
&\ge \frac{1}{2}\left(\|u\|^2 + \|v\|^2 - \frac{1}{2}\left(\|u\|^4_{L^4} + \|v\|^4_{L^4}\right)\right) \\
&\ge \frac{1}{2}\left(\|u\|^2 + \|v\|^2 - C\left(\|u\|^4 + \|v\|^4\right)\right)
\end{aligned}
$$

Hence, (ii) is satisfied. Now, we consider u and v such that $u|v| \neq 0$ and we fix $\|v\| = 1$. We let t_n be a increasing divergent sequence. Two cases can occur: either $\frac{\|g(t_n u, t_n v)\|}{t_n} \to \infty$ or $\frac{\|g(t_n u, t_n v)\|}{t_n} \to a \ge 0$. In the first case, we have

$$
\begin{aligned}
2F(t_n u, t_n v) &= t_n^2 \|u\|^2 + t_n^2 \|v\|^2 - \|g(t_n u, t_n v)\|^2 - t_n^2 \int_{-T}^{T} u^2 |t_n v + g(t_n u, t_n v)|^2 \, dt \\
&\le t_n^2 \left(\|u\|^2 + \|v\|^2 - \frac{\|g(t_n u, t_n v)\|^2}{t_n^2}\right) \to -\infty.
\end{aligned}
$$

In the second case, we let $h_n = t_n v + g(t_n u, t_n v)$ and we denote by w the weak limit of $w_n = \frac{h_n}{\|h_n\|}$; we also notice that

$$
\langle w_n, v \rangle = \frac{t_n}{\|h_n\|} \to (1+a)^{-\frac{1}{2}}.
$$

So $w = (1+a)^{-\frac{1}{2}} v + w^-$. Then we have

$$
2F(t_n u, t_n v) = t_n^2 \|u\|^2 + t_n^2 \|v\|^2 - t_n^2 \int_{-T}^{T} u^2 |h_n|^2 \, dt \tag{15}
$$

$$
= t_n^2 \left(\|v\|^2 + \|u\|^2\right) - t_n^4 \frac{\|h_n\|^2}{t_n^2} \int_{-T}^{T} u^2 |w_n|^2 \, dt \tag{16}
$$

But, $\int_{-T}^{T} u^2 |w_n|^2 \, dt \to \int_{-T}^{T} u^2 |w|^2 \, dt$. So, in order to conclude, it is enough to show that $\int_{-T}^{T} u^2 |w|^2 \, dt \neq 0$. We recall that

$$
-\|g(t_n u, t_n v)\|^2 = t_n^2 \int_{-T}^{T} u^2 \langle t_n v + g(t_n u, t_n v), g(t_n u, t_n v)\rangle \, dt
$$

Hence,

$$
\begin{aligned}
-\frac{\|g(t_n u, t_n v)\|^2}{t_n^2} &= t_n^2 \|h_n\| \int_{-T}^{T} u^2 \left\langle w_n, \frac{g(t_n u, t_n v)}{t_n} \right\rangle \, dt \\
&= t_n^2 \frac{\|h_n\|}{t_n} \int_{-T}^{T} u^2 \left\langle w_n, \frac{g(t_n u, t_n v)}{t_n} \right\rangle \, dt. \qquad (17)
\end{aligned}
$$

Comparing both sides of the equality at the limit, we see that

$$
\lim_{n\to\infty} \int_{-T}^{T} u^2 \langle w_n, \frac{g(t_n u, t_n v)}{t_n} \rangle \, dt = 0.
$$

But, $\frac{g(t_n u, t_n v)}{t_n} = \frac{\|h_n\|}{t_n} w_n - v$, converges weakly in $H^{\frac{1}{2}}_{per}(T)$ to $(1+a)^{\frac{1}{2}} w^-$ and thus strongly in L^2. Therefore,

$$
\int_{-T}^{T} u^2 \langle w, w^- \rangle \, dt = 0.
$$

Hence, if $\int_{-T}^{T} u^2 |v|^2 \, dt \neq 0$, then $\int_{-T}^{T} u^2 |w|^2 \, dt \neq 0$.
Finally, in order to show that F satisfies the (PS) condition, we first claim that $\|\nabla F(u,v)\| = \|\nabla E(u, v + g(u,v))\|$. Indeed, we recall that

$$
\langle \nabla_z E(u, v + g(u,v)), w \rangle = 0, \forall w \in H^-.
$$

Hence,

$$
\begin{aligned}
\langle \nabla_u F(u,v), h \rangle &= \langle \nabla_u E(u, v + g(u,v)), h \rangle + \langle \nabla_z E(u, v + g(u,v)), \nabla_u g(u,v) \cdot h \rangle \qquad (18) \\
&= \langle \nabla_u E(u, v + g(u,v)), h \rangle
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\langle \nabla_v F(u,v), w \rangle &= \langle \nabla_z E(u, v + g(u,v)), w + \nabla_v g(u,v) \cdot w \rangle \qquad (19) \\
&= \langle \nabla_z E(u, v + g(u,v)), w \rangle
\end{aligned}
$$

and this proves the claim. Now, if (u_n, v_n) is a (PS) sequence for F, then $(u_n, v_n + g(u_n, v_n)$ is a (PS) sequence for E and using Lemma 3.1 we finish the proof of (iv). □

Using the mountain pass lemma, we know that F has a critical point. But, as discussed above, we see that $\|\nabla F(u,v)\| = \|\nabla E(u, v + g(u,v))\|$. So the critical points of F correspond to critical points of E. One can characterize this critical point as the minimum of E on the generalized Nehari manifold

$$
\mathcal{N} = \left\{ (u,z) \in H^1_{per}(T) \times H^{\frac{1}{2}}_{per}(T) \setminus \{(0,0)\}, \ \textit{satisfying} \ \ (*) \right\}.
$$

where

$$
(*) \qquad \begin{cases}
\displaystyle\int_{-T}^{T} |u'|^2 + \frac{1}{4} u^2 dt = \int_{-T}^{T} u^2 |z|^2 dt; \\[2ex]
\displaystyle\int_{-T}^{T} \langle Az, z \rangle dt = \int_{-T}^{T} u^2 |z|^2 dt \\[2ex]
P^-(Az - u^2 z) = 0
\end{cases}
$$

It is important now to study the dependence of this critical point on T. Rescaling the interval to $[-1,1]$, we get an energy of the form

$$E(u,z)=\frac{T}{2}\left(\int_{-1}^{1}\frac{1}{T^2}|u'(s)|^2+\frac{1}{4}u^2(s)\;ds+\int_{-1}^{1}\langle A_{\frac{1}{T}}z,z\rangle(s)\;ds-\int_{-1}^{1}u^2|z|^2\;ds\right)$$

where $A_{\frac{1}{T}}z=-\frac{1}{T}Jz'+JBz$. Setting $\varepsilon=\frac{1}{T}$, we define

$$E_\varepsilon(u,z)=\frac{1}{2\varepsilon}\left(\int_{-1}^{1}\varepsilon^2|u'(s)|^2+\frac{1}{4}u^2(s)\;ds+\int_{-1}^{1}\langle A_\varepsilon z,z\rangle(s)\;ds-\int_{-1}^{1}u^2|z|^2\;ds\right).$$

We see that the critical points correspond to solutions to the equation

$$\begin{cases}-\varepsilon^2u''+\frac{1}{4}u=u|z|^2\\ -\varepsilon Jz'+JBz=u^2z\end{cases}\tag{20}$$

We will use the rescaled norms

$$\|u\|^2_{1,2,\varepsilon}=\frac{1}{\varepsilon}\int_{[-1,1]}\varepsilon^2|u'|^2+\frac{1}{4}u^2\;dt,\quad \|v\|^2_{\frac{1}{2},2,\varepsilon}=\frac{1}{\varepsilon}\int_{[-1,1]}\left(|A_\varepsilon|^{\frac{1}{2}}|z|\right)^2\;dt$$

and finally $\|u\|^p_{L^p,\varepsilon}=\dfrac{1}{\varepsilon}\displaystyle\int_{[-1.1]}|u|^p\;dt.$

From now on, we will say that a sequence $(u_\varepsilon,z_\varepsilon)\in H^1_{per}(1)\times H^{\frac{1}{2}}_{per}(1)$ satisfies property (A), if there exist $0<c_1<c_2$ such that

$$c_1\le E_\varepsilon(u_\varepsilon,z_\varepsilon)\le c_2\quad\text{and}\quad\|\nabla E_\varepsilon(u_\varepsilon,z_\varepsilon)\|_\varepsilon\to 0\tag{A}$$

Proposition 3.2. *Let $(u_\varepsilon,z_\varepsilon)\in H^1_{per}(1)\times H^{\frac{1}{2}}_{per}(1)$ satisfying (A). Then:*

(i) $\|u_\varepsilon\|_{1,2,\varepsilon}$ *and* $\|z_\varepsilon\|_{\frac{1}{2},2,\varepsilon}$ *are bounded.*

(ii) $\|z^-_\varepsilon-g(u_\varepsilon,z^+_\varepsilon)\|_{\frac{1}{2},2,\varepsilon}\to 0.$

(iii) $\|\nabla F_\varepsilon(u_\varepsilon,z^+_\varepsilon)\|_\varepsilon\to 0.$

Proof. The first point is similar to the (PS) condition so we omit it. So we focus on the second and last point. We set

$$g_\varepsilon=g(u_\varepsilon,z^+_\varepsilon),\quad z_1=z^+_\varepsilon+g_\varepsilon,\quad z_2=z^-_\varepsilon-g_\varepsilon$$

so that $z_\varepsilon=z_1+z_2$ and $z_2\in H^-$. Then we recall that $\langle\nabla_zE_\varepsilon(u,z_1),z_2\rangle=0$. Hence,

$$-\langle g_\varepsilon,z_2\rangle-\frac{1}{\varepsilon}\int_{[-1,1]}u^2_\varepsilon\langle z_1,z_2\rangle\;dt=0.$$

On the other hand, since $\|\nabla E_\varepsilon(u_\varepsilon, z_\varepsilon)\| \to 0$, we have

$$\langle \nabla_z E_\varepsilon(u_\varepsilon, z_\varepsilon), z_2\rangle = -\langle z_\varepsilon^-, z_2\rangle - \frac{1}{\varepsilon}\int_{[-1,1]} u_\varepsilon^2 \langle z_\varepsilon, z_2\rangle \; dt = o(\|z_2\|_{\frac{1}{2},2,\varepsilon}).$$

By taking the difference, it leads to

$$\|z_2\|^2_{\frac{1}{2},2,\varepsilon} + \frac{1}{\varepsilon}\int_{[-1,1]} u_\varepsilon^3 |z_2|^2 \; dt = o(\|z_2\|_{\frac{1}{2},2,\varepsilon})$$

Thus

$$\|z_2\|_{\frac{1}{2},2,\varepsilon} \le o(\|\nabla E_\varepsilon(u_\varepsilon, z_\varepsilon)\|) = o(1),$$

which proves (ii). Regarding (iii), it follows from writing

$$\nabla F_\varepsilon(u_\varepsilon, z_\varepsilon^+) = \nabla E_\varepsilon(u_\varepsilon, z_1) = \nabla E_\varepsilon(u_\varepsilon, z_\varepsilon + z_2).$$

Expanding the last term and using $\nabla E(u_\varepsilon, z_\varepsilon) \to 0$ and $z_2 \to 0$, we have the desired result. $\square$

Lemma 3.3. *If $(u_\varepsilon, z_\varepsilon) \in H^1_{per}(1) \times H^{\frac{1}{2}}_{per}(1)$ satisfies (A), then there exist t_ε and s_ε such that $(t_\varepsilon u_\varepsilon, s_\varepsilon z_\varepsilon^+ + g(t_\varepsilon u_\varepsilon, s_\varepsilon z_\varepsilon^+) \in \mathcal{N}$. Moreover,*

$$(t_\varepsilon, s_\varepsilon) \to (1,1), \quad \text{as} \quad \varepsilon \to 0.$$

Proof. Let us consider the map $G : \mathbb{R} \times \mathbb{R} \times H^- \to \mathbb{R} \times \mathbb{R} \times H^-$ defined by

$$G(t,s,h) = \begin{pmatrix} \langle \nabla_u E_\varepsilon(tu_\varepsilon, sz_\varepsilon^+ + h), tu_\varepsilon\rangle \\ \langle \nabla_z E(tu_\varepsilon, s(z_\varepsilon^+ + h)), s(z_\varepsilon^+ + h)\rangle \\ P^-(A_\varepsilon(z_\varepsilon^+ + h) - t^2 u_\varepsilon^2 (z_\varepsilon^+ + h)) \end{pmatrix}.$$

Clearly, $G(t,s,h) = 0$ if and only if $(tu_\varepsilon, s(z_\varepsilon^+ + h)) \in \mathcal{N}$. We set

$$c_\varepsilon = \frac{1}{\varepsilon}\int_{[-1,1]} |u_\varepsilon|^2 |z_\varepsilon^+ + g_\varepsilon|^2 \; dt$$

and from condition (A) we can assume that $c_\varepsilon \to c_0 > 0$. Indeed, since $(u_\varepsilon, z_\varepsilon)$ satisfies (A), we have from Proposition 3.2 that $\|u_\varepsilon\|_{1,2,\varepsilon}$ and $\|z_\varepsilon\|_{\frac{1}{2},2,\varepsilon}$ are bounded. In particular,

$$\langle \nabla_u E_\varepsilon(u_\varepsilon, z_\varepsilon), u_\varepsilon\rangle = o(1) \text{ and } \langle \nabla_z E_\varepsilon(u_\varepsilon, z_\varepsilon), z_\varepsilon\rangle = o(1).$$

Hence

$$0 < c_1 \le E_\varepsilon(u_\varepsilon, z_\varepsilon) = \frac{1}{2\varepsilon}\int_{[-1,1]} |u_\varepsilon|^2 |z_\varepsilon|^2 \; dt + o(1).$$

On the other hand, again using Proposition 3.2, we have $\|z_\varepsilon^- - g(u_\varepsilon, z_\varepsilon^+)\|_{\frac{1}{2},2,\varepsilon} \to 0$. Therefore, we have

$$c_\varepsilon = \frac{1}{\varepsilon}\int_{[-1,1]} |u_\varepsilon|^2 |z_\varepsilon|^2 \; dt + o(1) \ge 2c_1 + o(1).$$

We then compute

$$K = DG(1,1,g(u_\varepsilon, z_\varepsilon^+)) = \begin{bmatrix} B_\varepsilon & C_1 \\ C_1^* & \mathcal{A} \end{bmatrix},$$

where we have denoted

$$\mathcal{A}\varphi = P^-(A_\varepsilon\varphi - |u_\varepsilon|^2\varphi),$$

which is an invertible operator on H^-,

$$B_\varepsilon = \begin{bmatrix} 2\langle \nabla_u E_\varepsilon(u_\varepsilon, z_\varepsilon^+ + g_\varepsilon), u_\varepsilon\rangle & -2c_\varepsilon \\ -2c_\varepsilon & 2\langle \nabla_u E_\varepsilon(u_\varepsilon, z_\varepsilon^+ + g_\varepsilon) \end{bmatrix} \to B_0 := \begin{bmatrix} 0 & -2c_0 \\ -2c_0 & 0 \end{bmatrix},$$

and finally,

$$C_1 = \begin{bmatrix} -2\frac{1}{\varepsilon}\int |u_\varepsilon|^2 \langle z_\varepsilon^+ + g_\varepsilon, \cdot\rangle \; dt \\ 0 \end{bmatrix}.$$

In particular, since B_0 is invertible and $C_1^* B_0^{-1} C_1 = 0$, we have for ε small enough that K is invertible and K^{-1} is bounded uniformly as $\varepsilon \to 0$. Hence, by the inverse function theorem, since $G(1,1,g) \to 0$ as ε goes to zero, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, there exists t_ε and s_ε so that

$$G(t_\varepsilon, s_\varepsilon, h) = 0.$$

Moreover, we see that $|t_\varepsilon - 1| + |s_\varepsilon - 1| \le O(\|\nabla E_\varepsilon(u_\varepsilon, z_\varepsilon)\|)$. □

Lemma 3.4. *If $(u_\varepsilon, z_\varepsilon) \in H^1_{per}(1) \times H^{\frac{1}{2}}_{per}(1)$ satisfies (A), then there exists $(\tilde{u}_\varepsilon, \tilde{z}_\varepsilon) \in \mathcal{N}$ so that*

$$E_\varepsilon(\tilde{u}_\varepsilon, \tilde{z}_\varepsilon) = E_\varepsilon(u_\varepsilon, z_\varepsilon) + o(1).$$

Proof. Let $\tilde{u}_\varepsilon = t_\varepsilon u_\varepsilon$ and $\tilde{z}_\varepsilon = s_\varepsilon(z_\varepsilon^+ + g(t_\varepsilon u_\varepsilon, z_\varepsilon^+))$. Then we have

$$\begin{aligned} E_\varepsilon(\tilde{u}_\varepsilon, \tilde{z}_\varepsilon) &= E_\varepsilon\Big(u_\varepsilon + (t_\varepsilon - 1)u_\varepsilon, z_\varepsilon + (s_\varepsilon - 1)z_\varepsilon^+ + g(t_\varepsilon u_\varepsilon, z_\varepsilon^+) - g(t_\varepsilon u_\varepsilon, z_\varepsilon^+) \\ &\qquad + g(t_\varepsilon u_\varepsilon, z_\varepsilon^+) - z_\varepsilon^- + (s_\varepsilon - 1) g(t_\varepsilon u_\varepsilon, z_\varepsilon^+)\Big)) \\ &= E_\varepsilon(u_\varepsilon, z_\varepsilon) + O(\|\nabla E_\varepsilon(u_\varepsilon, z_\varepsilon)\|^2) \end{aligned}$$

□

It is important to notice that if $(u_\varepsilon, z_\varepsilon)$ is the solution obtained from the min-max process (or minimization on $\mathcal{N}$), then there exists $c_0 > 0$ such that

$$\frac{1}{\varepsilon}\int_{[-1,1]} u_\varepsilon^2 |z_\varepsilon|^2 \; dt \ge c_0. \tag{21}$$

Indeed, we have

$$E_\varepsilon(u_\varepsilon, z_\varepsilon) \ge \sup_{t>0, s>0, w\in H^-} E(tu_\varepsilon, sz_\varepsilon^+ + w) = \max_{t>0,s>0}\Big(t^2 - Ct^4 + s^2 - Cs^4\Big) \ge c_0$$

Thus if we define δ_ε by

$$\delta_\varepsilon = \inf_{(u,z)\in\mathcal{N}} E_\varepsilon(u,z),$$

then

$$E_\varepsilon(u_\varepsilon, z_\varepsilon) \geq \delta_\varepsilon \geq c_0 > 0. \tag{22}$$

Now we want to find an upper bound for δ_ε, and in order to do that we need to construct a suitable sequence $(u_\varepsilon, z_\varepsilon)$. We consider then the limiting functional defined on $H^1(\mathbb{R};\mathbb{R}) \times H^{\frac{1}{2}}(\mathbb{R};\mathbb{C})$ by

$$E(u,z) = \frac{1}{2}\Big(\int_{\mathbb{R}} |u'|^2 + \frac{1}{4}u^2 + \langle Az, z\rangle - u^2|z|^2 \, dt \Big).$$

Its critical points satisfy the following Euler-Lagrange equation

$$\begin{cases} -u'' + \frac{1}{4}u = u|z|^2 \\ Az = u^2 z \end{cases} \tag{23}$$

We denote by $\mathcal{M}$ the set of the ground state solutions and we let

$$\delta_0 = \inf\{E(u,z), \nabla E(u,z) = 0\} = E(U,Z),$$

for $(U,Z) \in \mathcal{M}$.

Lemma 3.5. *Let $(U,Z) \in \mathcal{M}$. Then up to translation and scaling,*

$$U(t) = 2^{-\frac{1}{4}} \cosh^{-\frac{1}{2}}(t), \qquad Z(t) = \frac{3}{2\sqrt{2}} \cosh^{-\frac{3}{2}}(t) \begin{pmatrix} e^{-\frac{t}{2}} \\ e^{\frac{t}{2}} \end{pmatrix}.$$

Proof. We recall from [3] that all the ground state solutions of (2) are classified. Indeed, if (U,Ψ) is a ground state solution, then there exists a parallel spinor $\Phi_0 \in \Sigma\mathbb{R}^3$, $x_0 \in \mathbb{R}^3$ and $\lambda > 0$ such that

$$U(x) = \Big(\frac{2\lambda}{\lambda^2 + |x - x_0|^2}\Big)^{\frac{1}{2}}$$

and

$$\Psi(x) = \Big(\frac{2\lambda}{\lambda^2 + |x - x_0|^2}\Big)^{\frac{3}{2}} \Big(1 - (x - x_0)\cdot\Big)\Phi_0.$$

In particular, if $x_0 = 0$, we can see that U is radial and $\Psi \in E(\mathbb{R}^3)$. So any ground state solution satisfies our radial ansatz. Hence, after the change to cylindrical coordinates, we have that $(U,\Psi) \in \mathcal{M}$. Therefore the energy level δ_0 corresponds indeed to ground state solutions of (2), which finishes the proof. $\square$

Lemma 3.6. *Let $(U,Z) \in \mathcal{M}$ and $\beta \in C_c^\infty(-1,1)$ such that $\beta = 1$ on $\big[-\frac{1}{2}, \frac{1}{2}\big]$, we set*

$$\overline{u}_\varepsilon(t) = \beta(t) U\left(\frac{t}{\varepsilon}\right), \qquad \overline{z}_\varepsilon(t) = \beta(t) Z\left(\frac{t}{\varepsilon}\right).$$

We have

$$E_\varepsilon(\overline{u}_\varepsilon, \overline{z}_\varepsilon) \to \delta_0 \quad \text{and} \quad \nabla E_\varepsilon(\overline{u}_\varepsilon, \overline{z}_\varepsilon) \to 0$$

as $\varepsilon \to 0$.

Proof. First we have that

$$\begin{aligned}\nabla_u E_\varepsilon(\overline{u}_\varepsilon, \overline{z}_\varepsilon) &= -\beta(t)U''\left(\frac{t}{\varepsilon}\right) + 2\varepsilon\beta'(t)U'\left(\frac{t}{\varepsilon}\right) + \varepsilon^2\beta''(t)U\left(\frac{t}{\varepsilon}\right) + \\ &\quad + \frac{1}{4}\beta(t)U\left(\frac{t}{\varepsilon}\right) - \beta^3 U\left(\frac{t}{\varepsilon}\right)\left|Z\left(\frac{t}{\varepsilon}\right)\right|^2 \\ &= 2\varepsilon\beta'(t)U'\left(\frac{t}{\varepsilon}\right) + \varepsilon^2\beta''(t)U\left(\frac{t}{\varepsilon}\right) + \left(\beta(t) - \beta^3(t)\right)U\left(\frac{t}{\varepsilon}\right)\left|Z\left(\frac{t}{\varepsilon}\right)\right|^2\end{aligned}$$

Next we notice that

$$\frac{1}{\varepsilon}\int_{[-1,1]} \varepsilon^2\left|\beta'(t)U'\left(\frac{t}{\varepsilon}\right)\right|^2\, dt = \varepsilon^2\int_{[-\frac{1}{\varepsilon},\frac{1}{\varepsilon}]} |\beta'(\varepsilon t))U'(t)|^2\, dt \le C\varepsilon^2\int_{\mathbb{R}} |U'|^2\, dt \to 0.$$

Similarly,

$$\frac{1}{\varepsilon}\int_{[-1,1]} \varepsilon^4\left|\beta''(t)U\left(\frac{t}{\varepsilon}\right)\right|^2\, dt \le C\varepsilon^4\int_{\mathbb{R}} |U(t)|^2\, dt \to 0.$$

But for the last term, we have

$$\frac{1}{\varepsilon}\int_{[-1,1]} (\beta(t) - \beta^3(t))^2\left|U\left(\frac{t}{\varepsilon}\right)\right|^2\left|Z\left(\frac{t}{\varepsilon}\right)\right|^4\, dt \le C\int_{\frac{1}{2\varepsilon}\le|s|\le\frac{1}{\varepsilon}} U^2(s)|Z(s)|^4\, ds \to 0.$$

Similarly, we can show that $\nabla_z E_\varepsilon(\overline{u}_\varepsilon, \overline{z}_\varepsilon) \to 0$. Now, we can move to the energy part:

$$\begin{aligned}2E_\varepsilon(\overline{u}_\varepsilon, \overline{z}_\varepsilon) &= \frac{1}{\varepsilon}\int_{[-1,1]} \varepsilon^2\left|\beta'(t)U\left(\frac{t}{\varepsilon}\right) + \frac{1}{\varepsilon}\beta(t)U'\left(\frac{t}{\varepsilon}\right)\right|^2 + \frac{1}{4}\beta^2(t)U^2\left(\frac{t}{\varepsilon}\right) \\ &\quad + \left\langle \beta(t)J\left(Z'\left(\frac{t}{\varepsilon}\right) + BZ\left(\frac{t}{\varepsilon}\right)\right) + \varepsilon\beta'(t)JZ\left(\frac{t}{\varepsilon}\right), \beta(t)Z\left(\frac{t}{\varepsilon}\right)\right\rangle \\ &\quad - \beta^4(t)U^2\left(\frac{t}{\varepsilon}\right)\left|Z\left(\frac{t}{\varepsilon}\right)\right|^2\, dt \\ &= \int_{[-\frac{1}{\varepsilon},\frac{1}{\varepsilon}]} \beta^2(\varepsilon s)\left(|U'(s)|^2 + \frac{1}{4}U^2(s) + \langle AZ, Z\rangle - |U(s)|^2|Z(s)|^2\right)\, ds \\ &\quad + \int_{|s|\le\frac{1}{\varepsilon}} \varepsilon^2|\beta'|^2(\varepsilon s)|U(s)|^2 + 2\varepsilon\beta'(\varepsilon s)\beta(\varepsilon s)U'(s)U(s)\, ds + \\ &\quad + \int_{|s|\le\frac{1}{\varepsilon}} \varepsilon\beta'(\varepsilon s)\beta(\varepsilon s)\langle JZ, Z\rangle\, ds \\ &\quad + \int_{\frac{1}{2\varepsilon}\le|s|\le\frac{1}{\varepsilon}} \left(\beta^4(\varepsilon s) - \beta^2(\varepsilon s)\right)|U(s)|^2|Z(s)|^2\, ds \\ &= I + II + III\end{aligned}$$

By using the dominated convergence theorem, one sees that

$$I \to 2E(U, Z) = 2\delta_0, \text{ as } \varepsilon \to 0.$$

On the other hand

$$II \leq C\varepsilon \to 0.$$

So it remains to show that the third term also converges to zero. Indeed,

$$\int_{\frac{1}{2\varepsilon}\leq|s|\leq\frac{1}{\varepsilon}} \left(\beta^4(\varepsilon s) - \beta^2(\varepsilon s)\right) |U(s)|^2|Z(s)|^2 \, ds \leq C \int_{\frac{1}{2\varepsilon}\leq|s|} |U(s)|^2|Z(s)|^2 \, ds \to 0,$$

as $\varepsilon \to 0$, which finishes the proof of the Lemma. □

Now we get the following the upper bound for δ_ε

Lemma 3.7.

$$\limsup_{\varepsilon\to 0} \delta_\varepsilon \leq \delta_0.$$

Proof. From Lemma 3.6, we have that $(\overline{u}_\varepsilon, \overline{z}_\varepsilon)$ satisfies assumption (A). Hence, using Lemma 3.4, we have

$$E_\varepsilon(\overline{u}_\varepsilon, \overline{z}_\varepsilon) = E_\varepsilon(\tilde{u}_\varepsilon, \tilde{z}_\varepsilon) + o(1) \geq \delta_\varepsilon + o(1).$$

So the conclusion follows by taking the lim sup in both sides. □

We have also a lower bound for δ_ε

Lemma 3.8.

$$\liminf_{\varepsilon\to 0} \delta_\varepsilon \geq \delta_0.$$

Proof. Let $(u_\varepsilon, z_\varepsilon)$ a minimizer of E_ε on $\mathcal{N}$. Then, it is a solution to the system (20) and $E_\varepsilon(u_\varepsilon, z_\varepsilon) = \delta_\varepsilon$. We first claim that there exist $r_0 > 0$, $\kappa_1, \kappa_2 > 0$ and $y_\varepsilon \in [-1, 1]$, such that

$$\frac{1}{\varepsilon}\int_{|t-y_\varepsilon|\leq\varepsilon r_0} |u_\varepsilon|^2 dt > \kappa_1, \qquad \frac{1}{\varepsilon}\int_{|t-y_\varepsilon|\leq\varepsilon r_0} |z_\varepsilon|^2 dt > \kappa_2. \tag{24}$$

In order to prove this claim, we assume by contradiction that the previous inequalities are not correct. Without loss of generality, we can assume that the first inequality is not correct. Then, for every $r > 0$,

$$\lim_{\varepsilon\to 0}\left(\frac{1}{\varepsilon}\sup_{y\in[-1,1]}\int_{|t-y_\varepsilon|\leq 2\varepsilon r} |u_\varepsilon|^2 \, dt\right) = 0.$$

Now we take $\beta_{\varepsilon,y}$ a cut-off function in $(y - 2r\varepsilon, y + 2r\varepsilon)$ such that $\beta_{\varepsilon,y}(t) = 1$ for $|t - y| \leq r\varepsilon$, we have

$$\lim_{\varepsilon\to 0}\left(\frac{1}{\varepsilon}\sup_{y\in[-1,1]}\int_{[-1,1]} |\beta_{\varepsilon,y} u_\varepsilon|^2 dt\right) = 0.$$

This yields $\|u_\varepsilon\|_{L^q,\varepsilon} \to 0$ for all $2 \le q < \infty$. Indeed, let $p > q > 2$ and $s > 0$ so that $q = sp + 2(1-s)$. Then we have

$$\frac{1}{\varepsilon}\int_{[-1,1]} |\beta_{\varepsilon,y} u_\varepsilon|^q \, dt \le \left(\frac{1}{\varepsilon}\int_{[-1,1]} |\beta_{\varepsilon,y} u_\varepsilon|^2 \, dt\right)^{1-s} \left(\frac{1}{\varepsilon}\int_{[-1,1]} |\beta_{\varepsilon,y} u_\varepsilon|^p \, dt\right)^{s}$$
$$\le C \left(\frac{1}{\varepsilon}\int_{[-1,1]} |\beta_{\varepsilon,y} u_\varepsilon|^2 \, dt\right)^{1-s} \|\beta_{\varepsilon,y} u_\varepsilon\|_{1,2,\varepsilon}^s$$

Hence, if we cover $[-1,1]$ by subintervals of radius $r\varepsilon$ with each subinterval overlapping with at most two others, we get

$$\frac{1}{\varepsilon}\int_{[-1,1]} |u_\varepsilon|^q \, dt \le C_2 \left(\sup_{y\in[-1,1]} \frac{1}{\varepsilon}\int_{|t-y_\varepsilon|\le 2\varepsilon r} |u_\varepsilon|^2 \, dt\right)^{1-s} \|u_\varepsilon\|_{1,2,\varepsilon}^s.$$

Now, we use (21) and write

$$c_0 \le \frac{1}{\varepsilon}\int_{[-1,1]} u_\varepsilon^2 |z_\varepsilon|^2 \, dt \le \|u_\varepsilon\|_{L^4,\varepsilon}^2 \|z_\varepsilon\|_{L^4,\varepsilon}^2 \le C\|u_\varepsilon\|_{L^4,\varepsilon}^2 \|z_\varepsilon\|_{\frac{1}{2},2,\varepsilon}^2,$$

and the contradiction follows by passing to the limit; so the first claim is proved.
Next we set

$$U_\varepsilon(s) = u_\varepsilon(\varepsilon s + y_\varepsilon), \qquad Z_\varepsilon(s) = z_\varepsilon(\varepsilon s + y_\varepsilon).$$

We notice that $(U_\varepsilon, Z_\varepsilon)$ is bounded in

$$\left(H^1_{loc}(\mathbb{R};\mathbb{R}) \times H^{\frac{1}{2}}_{loc}(\mathbb{R};\mathbb{C})\right) \cap \left(L^p(\mathbb{R};\mathbb{R}) \times L^p(\mathbb{R};\mathbb{C})\right)$$

for all $p > 1$. So we can extract a convergent subsequence that converges strongly in L^p for all p and weakly in $H^1_{loc} \times H^{\frac{1}{2}}_{loc}$ to (U_0, Z_0). We take a test function $h \in H^1(\mathbb{R};\mathbb{R})$ that is compactly supported in $[-R, R]$ and we have

$$\int_{\mathbb{R}} \left[-U_\varepsilon'' + \frac{1}{4}U_\varepsilon - U_\varepsilon |Z_\varepsilon|^2\right] h \, ds = \frac{1}{\varepsilon}\int_{|t-y_\varepsilon|\le \varepsilon R} \left[-\varepsilon^2 u_\varepsilon'' + \frac{1}{4}u_\varepsilon - u_\varepsilon |z_\varepsilon|^2\right] \tilde{h} \, dt = 0,$$

where $\tilde{h} = h\left(\frac{t-y_\varepsilon}{\varepsilon}\right)$. Similarly, taking $\varphi \in H^{\frac{1}{2}}(\mathbb{R};\mathbb{C})$, compactly supported in $[-R, R]$ we have

$$\int_{\mathbb{R}} \langle AZ_\varepsilon - U_\varepsilon^2 Z_\varepsilon, \varphi\rangle \, ds = \frac{1}{\varepsilon}\int_{|t-y_\varepsilon|\le \varepsilon R} \langle A_\varepsilon z_\varepsilon - u_\varepsilon^2 z_\varepsilon, \tilde{\varphi}\rangle \, dt = 0,$$

where $\tilde{\varphi}(t) = \varphi\left(\frac{t-y_\varepsilon}{\varepsilon}\right)$. Hence, (U_0, Z_0) is a solution of (23). Moreover, from (24), we have

$$\int_{[-r_0,r_0]} |U_\varepsilon|^2 \, ds = \frac{1}{\varepsilon}\int_{|t-y_\varepsilon|\le \varepsilon r_0} |u_\varepsilon|^2 \, dt > \kappa_1 > 0.$$

and

$$\int_{[-r_0,r_0]} |Z_\varepsilon|^2 \, ds = \frac{1}{\varepsilon}\int_{|t-y_\varepsilon|\le \varepsilon r_0} |z_\varepsilon|^2 \, dt > \kappa_1 > 0.$$

Therefore,

$$\int_{[-r_0,r_0]} |U_0|^2 \ ds \neq 0, \qquad \int_{[-r_0,r_0]} |Z_0|^2 \ ds \neq 0.$$

Thus, (U_0, Z_0) is not trivial and

$$E(U_0, Z_0) \geq \delta_0.$$

On the other hand, we have

$$E_\varepsilon(u_\varepsilon, z_\varepsilon) = \frac{1}{2\varepsilon} \int_{|t-y_\varepsilon|\leq 1} |u_\varepsilon|^2 |z_\varepsilon|^2 dt = \frac{1}{2} \int_{|s|\leq \frac{1}{\varepsilon}} |U_\varepsilon|^2 |Z_\varepsilon|^2 ds$$

Therefore,

$$\liminf_{\varepsilon\to 0} \delta_\varepsilon = \liminf_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon, z_\varepsilon) = E(U_0, Z_0) \geq \delta_0.$$

□

Lemma 3.9. *There exists $\varepsilon_0 > 0$ such that for $\varepsilon < \varepsilon_0$ the ground state solution of* (20) *is not the constant one.*

Proof. This is easily verified. Indeed, if

$$u = 1, \qquad z = \begin{pmatrix} \pm\sqrt{\frac{1}{8}} \\ \pm\sqrt{\frac{1}{8}} \end{pmatrix},$$

then

$$E_\varepsilon(u, z) = \frac{1}{2\varepsilon} \int_{[-1,1]} |u|^2 |z|^2 \ dt = \frac{1}{4\varepsilon} \to \infty > \delta_0.$$

□

This finished the global picture of the hamiltonian system by finding a family of periodic solutions converging to the homoclinic orbit generated by the spherical solution.

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